Casio
Victoria University of Wellington
Senior Mathematics Competition
2017

Solutions

Preliminary round: Thursday 18th May 2017
Time allowed 90 minutes

Instructions: Each √ is one mark.
You must record each student's results on the marking slip provided (**not one of your own design**) and attach it to any papers sent on to the Regional marker.

Total: 54 marks

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Solutions SENIOR MATHEMATICS COMPETITION 2017

Question 1

(i) Ari, Bella and Carl are comparing their bank balances in dollars. The sum of Ari's and Bella's balances is 100. The sum of Bella's and Carl's balances is 200. The product of Ari's and Carl's balances is 60000. What are their bank balances?

Solution:

a + b =100 b + c = 200 ac = 60 000 So $b^2 - 300b - 40\ 000 = 0$

Thus, b = 400, a = -300, c = -200 √

AND

b = -100 a = 200 c = 300 √ Both for 2 marks. [2]

ii) Xavier, Yuan and Zoe decide to compare their bank balances too. The product of Xavier's and Yuan's balances is 200. The product of Yuan's and Zoe's balances is 600. The product of Zoe's and Xavier's balances is 300. What are their bank balances?

Solution:

xy = 200, yz = 600, xz = 300.

So, $x^2y^2z^2 = 36\ 000\ 000$ thus xyz = 6000 or -6000

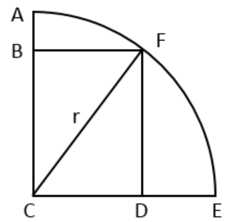
Divide by each of the 1st 3 equations in turn to give z = 30 x = 10 y = 20 √

AND

z = -30 x = -10 y = -20 √ Both for 2 marks. [2]

Question 2

A rectangle is drawn inside a quarter circle as shown. The distance from C to D is *12n* metres and the distance from D to E is *8n* metres, for some positive integer *n*. What is the distance from A to B in terms of *n*?



Solution:

r = 12n + 8n = 20n

BC = $\sqrt{(20n)^2 - (12n)^2} = \sqrt{256n^2}$ = 16n

Hence AB = 20 - 16n = 4n √ [1]

Question 3

Simplify this expression, by expressing x as a power of $\frac{1}{2}$ $\quad x = \sqrt{\frac{1}{2}\sqrt{\frac{1}{2}\sqrt{\frac{1}{2}}}}$

Solution:

$(\frac{1}{2}(\frac{1}{2}(\frac{1}{2})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}} = (\frac{1}{2}((\frac{1}{2})^{\frac{3}{2}})^{\frac{1}{2}})^{\frac{1}{2}} = (\frac{1}{2}(\frac{1}{2})^{\frac{3}{4}})^{\frac{1}{2}} = ((\frac{1}{2})^{\frac{7}{4}})^{\frac{1}{2}}$ So $x = (\frac{1}{2})^{\frac{7}{8}}$ √ [1]

Question 4

Consider a grid, with points *(x,y)* where both co-ordinates *x* and *y* are integers. We will assign values to some of the points as follows:

Firstly, 100 integers whose sum is 2016 are placed on any 100 of the points in the column with $x = 0$.

Next, assuming some numbers have been placed on the column with $x = n$, on the next column, where $x = n+1$, at each point $(n+1, m)$ if there are any numbers on any of (n, m) or $(n, m-1)$ or $(n, m+1)$ then put their sum at $(n+1, m)$.

Find the formula for the sum of the numbers placed on any column with $x = n$.

Solution:

Let S_n denote the sum of the numbers on column x = n.

The number on point (n,m) will contribute 3 times to S_{n+1} √

Hence $S_{n+1} = 3S_n$. Since $S_0 = 2016$ then $S_1 = 3S_0 = 2016 \times 3$

$S_2 = 3S_1 = 2016 \times 3^2$ etc Thus $S_n = 2016 \times 3^n$ √ [2]

Question 5

Let *p* be a prime number other than 3. Prove that $p^2 - 1$ is divisible by 3.

Solution:

Either p = 3k + 1 or p = 3k + 2, for some integer k, because p cannot have 3 as a divisor.

If p = 3k + 1 then $p^2 - 1 = (9k^2 + 6k + 1) - 1 = 3(3k^2 + 2k)$ which is divisible by 3. √

Similarly, if p = 3k+ 2 then $p^2 - 1 = (9k^2 + 12k + 4) - 1 = 3(3k^2 + 4k + 1)$ which is divisible by 3. √ [2]

Question 6

Given real numbers *a, b, c* that satisfy $0 = a + b + c = a^3 + b^3 + c^3$ show that at least one of *a, b, c* is zero.

Solution:

$a + b = -c$ and $a^3 + b^3 = -c^3 = (-c)^3 = (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$ √

Hence $3a^2b + 3ab^2 = 0$ Thus $0 = 3ab(a + b) = -3abc$ So $abc = 0$.

Thus at least one of a, b, or c must be 0 √ [2]

Question 7

Prove the following inequality for all real numbers $a,b \geq 1$

$$a\sqrt{b-1} + b\sqrt{a-1} \leq ab$$

Solution:

Let $x = \sqrt{a-1}$ and $y = \sqrt{b-1}$ then $a = x^2 + 1$ and $b = y^2 + 1$

We need to prove that $(x^2+1)y + (y^2+1)x \leq (x^2+1)(y^2+1)$ √

Now, $x^2 + 1 - 2x = (x-1)^2 \geq 0$ hence $x^2 + 1 \geq 2x$ so $x \leq \frac{x^2+1}{2}$ √

Thus $(y^2+1)x \leq \frac{1}{2}(y^2+1)(x^2+1)$ √

Similarly $(x^2+1)y \leq \frac{1}{2}(x^2+1)(y^2+1)$

By adding these two inequalities we have:

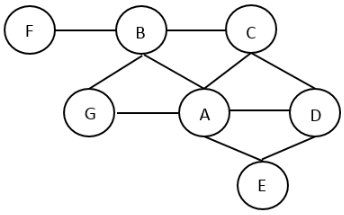
$(x^2+1)y + (y^2+1)x \leq (x^2+1)(y^2+1)$ √ [4]

Question 8

Complete this network diagram, showing friendships between a group of students, using the following clues:

Ben and Clare are friends; Erin and Clare are not friends; Ben is Fred's only friend; Aroha has the most friends; Dominic has 3 friends; Gino and Dominic are not friends; Erin has 2 friends.

Solution:



√ [1]

Question 9

Find, with reasoning, a four digit number *n* between 1000 and 9999 that satisfies both of these properties:

(a) The digits of *n* are increasing from left to right

(b) The sum *p + q* takes the smallest possible value, where *p* is the ratio of the thousands digit by the hundreds digit and *q* is the ratio of the tens digit by the ones digit.

Solution:

Let our number be *abcd*, where $1 \leq a < b < c < d \leq 9$ *

and we need to minimize $p + q = \frac{a}{b} + \frac{c}{d}$ ** √

If we reduce 'a' and increase 'd' while keeping 'b' and 'c' fixed we shall make ** smaller and still satisfy *. Thus we have a = 1 and b = 9 √

$c \geq b + 1$ Let $c = b + 1$ to minimise ** then we have:

$p + q = \frac{1}{b} + \frac{b+1}{9} = \frac{1}{b} + \frac{1}{9} + \frac{b}{9} = \frac{9+b^2}{9b} + \frac{1}{9}$ √

As $9 + b^2 = (3 - b)^2 + 6b \geq 6b$ then we have $p + q = \frac{9+b^2}{9b} + \frac{1}{9} \gg \frac{6b}{9b} + \frac{1}{9} = \frac{7}{9}$

This is minimal when $(3 - b)^2 = 0$ i.e. b = 3. Hence abcd = 1349 √ [4]

Question 10

A right-angled triangle with hypotenuse h has perimeter P. Express the area of the triangle in terms of h and P, in its simplest form.

Solution:

Let the 2 smaller sides be '*a*' and '*b*'. Then $P = a + b + h$ and $h^2 = a^2 + b^2$

Hence $(P - h)^2 = (a + b)^2 = h^2 + 2ab$ √

Thus $2ab = (P - h)^2 - h^2$

Now $A = \frac{1}{2}ab$ so A = $\frac{(P-h)^2 - h^2}{4} = \frac{P^2 - 2Ph}{4}$ √ [2]

Question 11

Find all triples of non-zero integers whose sum and product are equal, showing clearly that you have all of them.

Solution:

Let a, b, c be nonzero integers with $a + b + c = abc$ * WLOG suppose $|a| \leq |b| \leq |c|$ **

Then $|abc| = |a + b + c| \leq |a| + |b| + |c| \ll 3|c|$

Since $c \neq 0$ $then$ $|ab| \leq 3$ thus $|ab| = 1$ or 2 or 3. **√**

If $|ab| = 1$ then either $ab = 1$ $giving$ $a = b = \pm 1,$ $hence * gives$ $2a + c = c, ie$ $a = 0$ a contradiction or $ab = -1$ $giving$ $a = \pm 1$ and $b = \mp 1,$ $hence * gives$ $c = -c, ie$ $c = 0,$ a contradiction. **√**

If $|ab| = 2$ then either $ab = 2$ $giving$ $a = \pm 1$ and $b = \pm 2$ $since$ $**,$ $hence * gives$ $3a + c = 2c, ie$ $c = 3a = \pm 3$ ie (a,b,c) = (1,2,3) or (-1, -2, -3) **√**

or $ab = -2$ $giving$ $a = \pm 1$ and $b = \mp 2,$ $hence * gives - a + c = -2c, ie$ $3c = \pm 1$ $which$ has no $integer$ $solution.$ **√**

If $|ab| = 3$ then either $ab = 3$ $giving$ $a = \pm 1$ and $b = \pm 3$ $since$ $**,$ $hence * gives$ $4a + c = 3c, ie$ $2c = \pm 4, giving$ $c = \pm 2$ $which$ $contradicts$ $**$

or $ab = -3$ $giving$ $a = \pm 1$ and $b = \mp 3,$ $hence * gives - 2a + c = -3c, ie$ $4c = \pm 2$ $which$ has no $integer$ $solution.$ **√**

Hence the only triples of nonzero integers with sum and product equal are:

(1,2,3) and (-1,-2,-3) **√** [**6**]

Question 12

Tamara cycled from her place to Penny's. Penny left home **at the same time** as Tamara and drove from her place to Tamara's. They passed each other outside Cara's cafe at midday. They kept going, with Tamara reaching Penny's place at 1:20pm and Penny reaching Tamara's place at 12:20pm. Both had constant speeds throughout the journey. At what time did they leave home?

Solution:

Let Cara's café be x km from Tamara's place and y km from Penny's place, and let the time taken to get to Cara's be t hours. Then Tamara's speed is $\frac{x}{t}$ and Penny's is $\frac{y}{t}$.

It takes Penny $\frac{1}{3} = x \div \frac{y}{t} = \frac{tx}{y}$ of an hour to get to Tamara's from Cara's.

It takes Tamara $\frac{4}{3} = y \div \frac{x}{t} = \frac{ty}{x}$ of an hour to get to Penny's from Cara's. **√**

Dividing these two equations gives $4 = \frac{y^2}{x^2}$ ie $2x = y$ **√**

so the first part of the trip took $t = \frac{4}{3} \div 2 = \frac{2}{3} hour.$

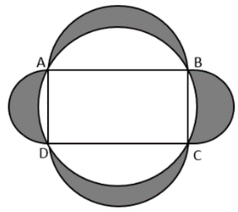
Hence they left home at 11:20am. **√** [3]

Question 13
Given 1009 different natural numbers not exceeding 2016, prove that there are always two of them whose sum is 2017.
Solution:

Assume that no two of the numbers sum to 2017. Then our 1009 numbers subtracted from 2017 each have a difference not equal to any of our other numbers. √

Hence we have 1009 x 2 distinct natural numbers not exceeding 2016, but 1009 x 2 = 2018, so we have a contradiction. √ [2]

Question 14
A rectangle has half circles on each side and a circle is drawn through the 4 vertices of the rectangle. Show with full reasoning that the area of the rectangle is equal in size to the total area of the 4 shaded crescents.



Solution:

Let AB = 2x and BC = 2y Then Area of rectangle $A_r = 4xy$

Area of circle = $A_c = \pi(x^2 + y^2)$ *because* $r = \sqrt{(x^2 + y^2)}$

Area of half circles with diameter AB and DC is $A_{AB+DC} = \pi x^2$ and the area of half circles with diameter BC and AD is $A_{BC+AD} = \pi y^2$ √

Outer circle area = Circle – rectangle = $A_c - A_r = \pi(x^2 + y^2) - 4xy$ √

Crescents area = half circles – outer circle area

= $\pi x^2 + \pi y^2 - (\pi(x^2 + y^2) - 4xy) = 4xy$, *the area of the rectangle.* √ [3]

Question 15
(i) Show that a natural number and the sum of its digits have the same remainder when divided by 9.
Solution:

A natural number *A natural number a may be written as* $d_n d_{n-1} \dots d_1 d_0$ that is $a = d_n \times 10^n + d_{n-1} \times 10^{n-1} + \dots + d_1 \times 10 + d_0$ √

= $(d_n + d_{n-1} + \dots + d_1 + d_0) + d_n \times 99\dots9 + d_{n-1} \times 99\dots9 + \dots + d_1 \times 9$ where each number 99…9 multiplied by d_m has m digits. Hence a and the sum of its digits have the same remainder when divided by 9. √ [2]

(ii) Let $S_0 = 2^{2016}$ and denote by S_1 the sum of the digits of S_0, by S_2 the sum of the digits of S_1, by S_3 the sum of the digits of S_2, and by S_4 the sum of the digits of S_3. Find S_4.
Solution:

Note that $S_0 = 2^{2016} = (2^3)^{672} = 8^{672} < 10^{672}$ Thus S_0 has at most 672 digits and hence $S_1 \leq 672 \times 9 < 1\,000 \times 10 = 10\,000$ √

Thus S_1 has at most 4 digits and hence $S_2 \leq 4 \times 9 = 36$ Thus either S_2 has only 1 digit or it has two with the first < 4. Hence $S_3 \leq 4 + 9 = 13$ This shows that $S_4 < 10$. √

Next we see that $S_0 = 2^{2016} = (2^6)^{336} = 64^{336} = (9 \times 7 + 1)^{336}$ which has remainder 1 when divided by 9. √ By part (i) so does S_1, and then S_2, S_3 and S_4. The only natural number <10 that has remainder 1 when divided by 9 is 1. Hence $S_4 = 1$ √ [4]

Question 16

(i) Without using a calculator find x in its simplest form.

$$x = \sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}$$

Solution:

$x^2 = 7+4\sqrt{3} + 7-4\sqrt{3} + 2\sqrt{7+4\sqrt{3}}\sqrt{7-4\sqrt{3}}$ √

$= 14 + 2\sqrt{(49-16\times 3)} = 16$ √

hence $x = 4$ √

[OR may use $(a+b)^2 = a^2+b^2+2ab$, with $a = 2 \; and \; b = \sqrt{3}$ then $\sqrt{7+4\sqrt{3}} =$

$\sqrt{2^2 + (\sqrt{3})^2 + 2\times 2\sqrt{3}} = \sqrt{(2+\sqrt{3})^2} = 2+\sqrt{3}$ √

$and\ similarly$ $\sqrt{7-4\sqrt{3}} = \sqrt{(2-\sqrt{3})^2} = 2-\sqrt{3}$ √

Thus $x = \sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}} = 2+\sqrt{3}+2-\sqrt{3} = 4$ √] [3]

(ii) Express $y = \frac{1}{\sqrt[3]{3}-\sqrt[3]{2}}$ in the form $y = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ with *a, b, c* integers

Solution:

Using $a^3 - b^3 = (a-b)(a^2+ab+b^2)\ gives\ us\ \frac{1}{a-b} = \frac{a^2+ab+b^2}{a^3-b^3}$ √

Let $a = \sqrt[3]{3}\ and\ b = \sqrt[3]{2}\ \ then\ \ y = \frac{1}{a-b} = \frac{(\sqrt[3]{3})^2+\sqrt[3]{3}\sqrt[3]{2}+\sqrt[3]{2}^2}{3-2}$ √

$= \sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$ √ [3]

Question 17

Let *a, b* > 0 be real constants. Solve this system of equations for x and y in terms of *a* and *b*.

$$x\left(1+\frac{x}{y}\right) = a^2b \qquad\qquad y\left(1+\frac{y}{x}\right) = b$$

Solution:

$We\ can\ rewrite\ these\ as$ $\frac{x}{y}(x+y) = a^2b$ and $\frac{y}{x}(x+y) = b$ √

Dividing one by the other gives us $\frac{x^2}{y^2} = a^2$ $so\ either$ $\frac{x}{y} = a$ or $\frac{x}{y} = -a$ √

$If\ \frac{x}{y} = a\ \ then\ \ x = ay$ Substitute into 1st equation gives us $a(ay+y) = a^2b$

$ie\ \ y(a+1) = ab$ thus $y = \frac{ab}{a+1}$ $and\ x = ay = \frac{a^2b}{a+1}$ √

$If\ \frac{x}{y} = -a\ \ then\ \ x = -ay$ Substitute into 1st equation gives:

$-a(-ay+y) = a^2b\ \ i.e.\ \ y(a-1) = ab$ √

thus $y = \frac{ab}{a-1}$ $and\ x = -ay = \frac{-a^2b}{a-1} = \frac{a^2b}{1-a}$ √ [5]