and

SENIOR MATHEMATICS COMPETITION 2009

Solutions

Preliminary round
Friday 22nd May 2009
Time allowed 1 ½ hours

It is not possible to provide all possible methods students will use to solve these questions.
If an answer is correct award full marks.

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and

1. What is the units digit, ie the last digit, in

$(1!)^3 + (2!)^3 + (3!)^3 + \dots\dots + (2008!)^3 + (2009!)^3$
$= 1^3 + 2^3 + 6^3 + 24^3 + 120^3 + 540^3 + \dots + (2009!)^3$ ✔

Taking last digit only

$1 + 8 + 6 + 4 + 0 + 0 + 0 + \dots\dots + 0$ ✔

$= 9$ ✔ [3]

last digit is 9

2. How many different values can the angle ABC take, where A, B and C are distinct vertices of a cube.

In a unit cube, there are 3 different types of triangles, with lengths 1, 1, √2 ; 1, √2, √3 and √2,√2, √2 ✔

The 1, 1, √2 has ∠ABC = 45° or 90°
The 1, √2, √3 has ∠ABC = 90° or 35.26° or 54.74°
The √2,√2, √2 has ∠ABC = 60° } ✔

hence there are 5 different values for ∠ABC. ✔ [3]

3. Evaluate

$$\sqrt[3]{\frac{1\times2\times4+2\times4\times8+3\times6\times12+\dots}{1\times3\times9+2\times6\times18+3\times9\times27+\dots}}$$ ✔

$$=\sqrt[3]{\frac{(1\times2\times4)(1+8+27+\dots}{(1\times3\times9)(1+8+27+\dots}}$$ ✔

$$=\sqrt[3]{\frac{8}{27}}$$ ✔ [3]

$$=\frac{2}{3}$$

4. Solve for *a* and *b* in log 96 = *a* log 24 + *b* log 36 [4]

$96 = 24^a\, 36^b$

$2^5 . 3 = (2^3\ 3)^a (2^2 . 3^2)^b$ ✔

$2^5 . 3 = 2^{3a+2b}\ 3^{a+2b}$ ✔

Giving
$5 = 3a + 2b$
$1 = a + 2b$ } ✔ $a = 2$ and $b = -½$ ✔

5. If the number of sides of a regular polygon is multiplied by five, the sum of the interior angles of the new polygon is a multiple of the sum of the interior angles of the original polygon.

For which regular polygon does this occur ?

The interior sum of angles is $(n - 2) \times 180°$
Sum of interior angles of new polygon = $k \times$sum of int $\angle$'s of original polygon $k \in \mathbb{R}$

$$(5n - 2) \times 180° = k \times (n - 2) \times 180° \checkmark$$

$$k = \frac{5n - 2}{n - 2}$$

$$k = 5 + \frac{8}{n - 2} \checkmark$$

To be a multiple $\frac{8}{n-2}$ must be a natural number

i.e $n - 2$ must be 1, 2, 4 or 8 ✓

$n = 3,\ k = 13$
$n = 5,\ k = 9$
$n = 6,\ k = 7$
$n = 10,\ k = 1$ ✓

polygons are the triangle, square, hexagon and decagon ✓ [5]

6. Function $f(x)$ obeys the rule $f(x+1) = f(x) - f(x-1)$.

Given $f(2) = 5$ and $f(1) = 2$, find $f(2009)$. [6]

$f(3) = f(2) - f(1) = 3$
$f(4) = f(3) - f(2) = -2$
$f(5) = -5$
$f(6) = -3$
$f(7) = 2 \quad (= f(1))$
$f(8) = 5 \quad (= f(2))$

✓ ✓ ✓ reward progress

The function $f(x)$ is periodic , with period 6. ✓

2009 mod 6 = 5 (Since 2009 = 6 × 334 + 5)✓

$f(2009) = f(5) = -5$ ✓

7. A bin contains 20 balls: 8 red, 7 black and 5 blue .
We draw three balls at random (without replacement) from the bin and we say we "win" if our three balls have only two colours. (i.e we "win" if we draw two balls of one colour and another ball of a different colour.)

What is the probability of winning this particular game?

Number of possible outcomes of drawing 3 balls from 20 is $^{20}C_3 = 1140$. ✔

Need to find how many of the possibilities can win the game.
Need to draw (i) exactly 2 red and one other colour
i.e $^8C_2 \times 12 = 336$ ✔

(ii) exactly 2 black and one other colour
$^7C_2 \times 13 = 273$ ✔

(iii) exactly 2 blue and one other colour
$^5C_2 \times 15 = 150$ ✔

Solution 2

P(2 reds only) =P(RRR′)+P(RR′ R)+P(R′ RR)

$= 3 \times \frac{8}{20} \times \frac{7}{19} \times \frac{12}{18}$ ✔

$= \frac{2016}{6840}$ ✔

P(2 black only) = $\frac{1638}{6840}$ ✔

P(2 blue only) = $\frac{900}{6840}$ ✔

P(winning) = $\frac{4554}{6840} = \frac{253}{380}$ ✔

Total number of winning choices = 336 + 273 + 150 = 759
so, the probability of winning is $\frac{759}{1140} = \frac{253}{380}$ ✔ [5]

(= 0.666)

8. Two real numbers, x and y, satisfy the condition $x + y = 2$.

Show $\boldsymbol{xy(x^2 + y^2) \le 2}$. [5]

Solution 1

$= xy((x+y)^2 - 2xy)$ ✔

$= xy(4 - 2xy)$ ✔

$= 2(2xy - x^2y^2)$ ✔

$= 2(1 - (1 - xy)^2)$ ✔

$\le 2 \times 1$ **since** $(1 - xy)^2 \ge 0$ ✔

≤ 2

Solution 2

$(xy - 1)^2 \ge 0$ ✔

$x^2y^2 - 2xy + 1 \ge 0$

$xy(xy - 2) + 1 \ge 0$

$xy(2 - xy) \le 1$ ✔

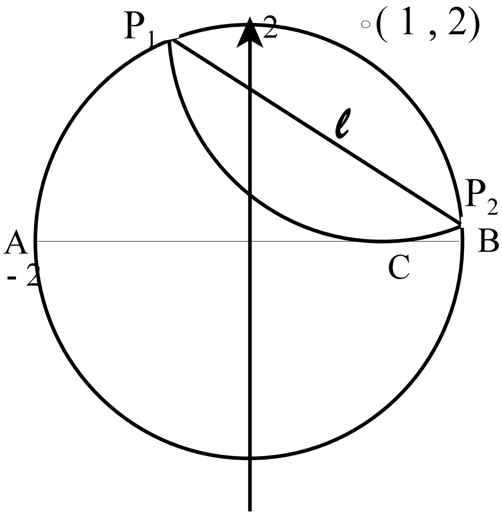
$xy(4 - 2xy) \le 2$ ✔

$xy(4 + x^2 + y^2 - (x+y)^2) \le 2$ ✔

$xy(4 + x^2 + y^2 - 4) \le 2$ ✔

$xy(x^2 + y^2) \le 2$

9.



A segment of a circle is folded so that its arc is tangent to the diameter AB, at point C .

The diameter is divided so that the ratio **AC : CB = 3:1.**

Calculate the length of the chord , ℓ .

[6]

Set up a coordinate system of a circle with origin as the centre and a radius of 2 units
The circle has equation $x^2 + y^2 = 4$
The segments are tangential to AB, so the centre of the folded segments circle is at (1 , 2) since C is at (1, 0)
This circle has equation $(x - 1)^2 + (y - 2)^2 = 4$ ✔

Let P_1 and P_2 be the points on the intersection of these two circles

$$(x - 1)^2 + (y - 2)^2 = x^2 + y^2$$
$$\Rightarrow \quad y = {}^{(5 - 2x)}/_4 \quad ✔$$

Substituting $x^2 + [{}^{(5 - 2x)}/_4]^2 = 4$

$$20x^2 - 20x - 39 = 0 \quad ✔$$
$$x^2 - x = 1.95$$
$$x = 0.5 \pm \sqrt{2{\cdot}2} \quad ✔ \; x = -0.98324 \text{ or } 1.98324$$

$$y = {}^{(19 \mp 2\sqrt{2{\cdot}2})}/_4$$

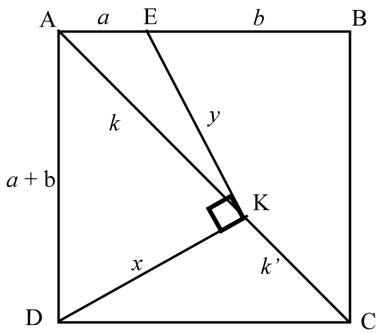
Use Pythagoras P_1 $(0.5 - \sqrt{2{\cdot}2}, {}^{(19 + 2\sqrt{2{\cdot}2})}/_4)$ P_2 $(0.5 + \sqrt{2{\cdot}2}, {}^{(19 - 2\sqrt{2{\cdot}2})}/_4)$

P_1 (-0.98324, 1.74162) P_2 (1.98324, 0.2584) ✔

$$\text{Distance} = \sqrt{\left(2\sqrt{2.2}\right)^2 + \left(\sqrt{2.2}\right)^2}$$
$$= \sqrt{11}$$
$$= 3.31662 \quad \text{Units} \quad ✔$$

$$\text{Or distance} = \sqrt{(1.98324 + 0.98324)^2 + (1.74162 - 0.2584)^2}$$
$$= 3.316$$

10.



ABCD is a square with E a point on AB such that AE : EB = a:b .
K is a point such that angle DKE = 90º .

Find the ratio AK : KC in terms of a and b.

[10]

Label the lengths as shown in the diagram AK:KC = k: k'

In △ AKE $y^2 = a^2 + k^2 - 2ak\cos 45$

$y^2 = a^2 + k^2 - \sqrt{2}ak$ ---① ✔

In △ AKD $x^2 = (a + b)^2 + k^2 - \sqrt{2}(a + b)k$ ---② ✔

In △DKE $DE^2 = x^2 + y^2$

In △DAE $DE^2 = a^2 + (a + b)^2$

$x^2 + y^2 = a^2 + (a + b)^2$ ---③ ✔✔

① + ② $a^2 + k^2 - \sqrt{2}ak + (a + b)^2 + k^2 - \sqrt{2}(a + b)k = a^2 + b^2$

$\Rightarrow 2k^2 - \sqrt{2}(a + b)k - \sqrt{2}ak = 0$

$k(2k - \sqrt{2}(a + b) - \sqrt{2}a) = 0$ ✔

$k(2k - \sqrt{2}(2a + b)) = 0$

Since $k \neq 0$ $k = {}^{\sqrt{2}(2a + b)}/_2$ ---④ ✔

$AC^2 = 2(a + b)^2$

$k + k' = \sqrt{2}(a + b)$

$k' = \sqrt{2}(a + b) - {}^{\sqrt{2}(2a + b)}/_2$

$k' = {}^{\sqrt{2}b}/_2$ ✔✔

AK : KC = k: k'

$= {}^{\sqrt{2}(2a + b)}/_2 : {}^{\sqrt{2}b}/_2$

$= 2a + b : b$ ✔✔