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SENIOR MATHEMATICS COMPETITION 2010

Solutions

Preliminary round
Thursday 20th May 2010
Time allowed 1 ½ hours

It is not possible to provide all possible methods students will use to solve these questions.
If an answer is correct award full marks.

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NZAMT and CASIO

SENIOR MATHEMATICS COMPETITION **Solutions** 2010

1. Finding the value of n.

$$(10^{2010} + 25)^2 - (10^{2010} - 25)^2 = (10)^n$$

LHS $= (2 \times 10^{2010})(50) \times 10^{10}$ ✔✔

$= 10^{2012}$ ✔

Thus $n = 2012$ ✔ [4]

2. Possible combinations of digits summing to 12

a) $3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 = 12$
b) $2^2 + 2^2 + 2^2 + 0^2 + 0^2 + 0^2 = 12$ ✔✔
c) $2^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 = 12$

For a 3, 3 1's and 2 of 0's if first is 3, then no. of ways $= \dfrac{5!}{3!2!} = 10$

For 3, 3 1's and 2 of 0's if first is 1, then no. of ways $= \dfrac{5!}{2!\,2!} = 30$

✔✔

For 3 2's and 3 of 0's first must be 2, then no. of ways $= \dfrac{5!}{2!\,3!} = 10$ ✔✔

for all these ways

For 2 2's and 4 of 1's if first is 2 then no of ways $= \dfrac{5!}{4!} = 5$

For 2 2's and 4 of 1's if first is 1 then no of ways $= \dfrac{5!}{2!\,3!} = 10$

Total number of ways $= 10 + 30 + 10 + 5 + 10$
$= 65$ ✔ [7]

3. A is chosen first and the others relative to A around the circle can be in the order

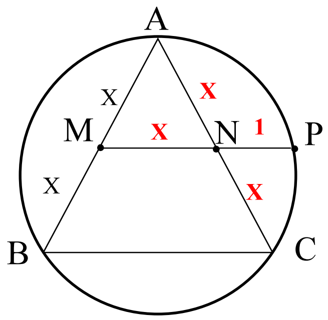
ABCD, ABDC, ACBD, ACDB, ADBC, ADCB ✔

for AB to intersect CD, B must be in third position ✔

i.e ACBD or ADBC

so Probability of intersection $= {}^{2}/_{6} = \frac{1}{3}$ ✔ [3]

4.



We require ratio of MN:NP
Let NP =1
and AM = x ✔

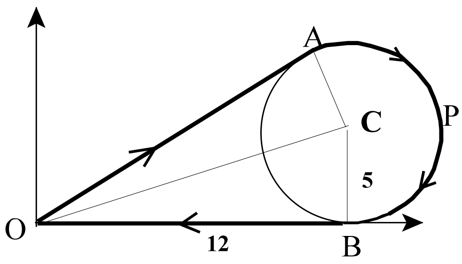
Using intersecting chord theorem
$x.x = 1.(x + 1)$ ✔

$x^2 - x - 1 = 0$

$x = \dfrac{1 + \sqrt{5}}{2}$ ✔

Ratio MN:NP is $\dfrac{1+\sqrt{5}}{2} : 1$ This is the Golden ratio. (It cannot be negative) [3]

5.



The centre of the circle is at (12, 5)

OA and OB are tangents of length 12 to the circle. ✔

tan OCB = $^{12}/_{5}$
OCB = 67.380° ✔
ACB =134.7602°

Arc APB = $\dfrac{(360^\circ - 134\cdot 7602^\circ)}{360^\circ} \times 2\pi \times 5$ ✔
= 19.66 ✔

Thus total length is 2×12 + 19.66 = 43.66km ✔

[5]

6. *Method 1 using change of base formula.*

$\log_8 a + \log_4 b^2 = 5$

$\dfrac{\log a}{3\log 2} + \dfrac{\log b}{\log 2} = 5$ ✔

$\log a + 3\log b = 15\log 2$

$ab^3 = 2^{15}$ ✔ (1)

$\log_8 b + \log_4 a^2 = 7$

$\dfrac{\log b}{3\log 2} + \dfrac{2\log a}{2\log 2} = 7$

$\log b + 3\log a = 21\log 2$

$a^3 b = 2^{21}$ - (2) ✔

Multiplying (1) and (2) gives $(ab)^4 = 2^{36}$ So that $ab = 2^9$
$b = 2^9 / a$ ✔

Substitute this into (2)

$a^3 \dfrac{2^9}{a} = 2^{21}$ - (2)

$a^2 = 2^{12}$

Thus $a = 2^6$
= 64 ✔ and b = 8 ✔ [6]

Method 2 using a substitution.

$\log_8 a + \log_4 b^2 = 5$

let $x = \log_8 a$

$a = 8^x$ ✔

$x + \log_4 (8^y)^2 = 5$

$x + \log_4 (4^{3y}) = 5$

$x + 3y \quad = 5$ ① ✔

$\log_8 b + \log_4 a^2 = 7$

let $y = \log_8 b$

$b = 8^y$ ✔

$y + \log_4 (8^x)^2 = 7$

$y + \log_4 (4^{3x}) = 7$

$y + 3x$ ② $= 7$ ✔

3 ① -② gives 8y = 8 so y=1 and thus x = 2 ✔

since $a = 8^x$ then a = 64 and since $b = 8^y$ then b = 8 ✔

7. ***f(n) = an + b*** *a,b*$\in \mathbb{I}$

Method 1

$\boldsymbol{f(3n+1)} = 3an + a + b$

$\boldsymbol{f(3n) + 1} = 3an \quad + b \quad + 1$

$\boldsymbol{3f(n) + 1} = 3an \quad + 3b + 1$ ✔

Since ***f(3n + 1)***, ***f(3n) + 1*** and ***3 f(n) + 1*** are consecutive integers in some order, by subtracting *3an +b* from above then so are a, 1, 2b+1 ✔

the possibilities are 1, 2, 3 or 0, 1, 2 or -1, 0, 1

for 1, 2, 3 ⇒ a = 2 and 2b + 1 = 3 ✔ then ***f(n) = 2n + 1***

for 0, 1, 2 2b + 1 = 0 or 2 so no solution is possible ✔

for -1, 0, 1 2b + 1 = -1 ⇒ a = 0 then ***f(n) = -1*** ✔

so only two solutions are possible ***f(n) = -1*** and ***f(n) = 2n + 1*** [5]

Method 2

In this order ***f(3n + 1), f(3n) + 1, 3 f(n) + 1***

3an+a+ b, 3an+b+1, 3an+3b+1 ✔

difference

1-a =1 and *2b=1* since consecutive.

not possible b≠ ½ ✔

or ***f(3n + 1), 3 f(n) + 1, f(3n) + 1***

3an+a+ b, 3an+3b+1, 3an+b+1,

2b-a+1 =1 and *-2b =1*

not possible b≠ -½

In this order ***f(3n) + 1, f(3n + 1),3 f(n) + 1***

3an+b+1, 3an+a+ b,3an+3b+1

difference

a-1 =1 and *2b-a+1 = 1*

a=2 and *b=1*

f(n) =2n +1 ✔

or ***f(3n) + 1, 3 f(n) + 1, f(3n + 1)***

3an+b+1,3an+3b+1 ,3an+a+ b

2b=1 and *a-2b-1=1*

not possible b≠ ½

In this order

3 f(n) + 1,f(3n) + 1, f(3n + 1)

3an+3b+1, 3an+b+1,3an+a+ b

-2b =1 and a-1 =1

not possible b≠ -½

3 f(n) + 1, f(3n + 1), f(3n) + 1

3an+3b+1,3an+a+ b, 3an+b+1

a-2b-1=1 and *1-a=*1

a=0 and *b=-1*

f(n) =-1 ✔

Thus only two functions are possible ***f(n) =2n +1*** or ***f(n) = -1*** ✔

8. $(x^2 - 3x + 1)^{x+1} = 1$

Case 1 base =1 exponent ≠0
$x^2 - 3x + 1 = 1$ so $x = 0$ or 3 ✔

Case 2 base ≠0 , exponent = 0
$x + 1 = 0$ so $x = -1$ ✔

Case 3 base = -1 and exponent is even
$x^2 - 3x + 1 = -1$
$x^2 - 3x + 2 = 0$
$(x - 1)(x-2) = 0$ ✔
$x = 1$ only (if $x = 2$ you have an odd exponent)✔

Solutions are $x = -1, 0, 1$ *and* 3 [4]

9. P = a(a + 2d) (a + 4d) Q = (a + d) (a + 3d)

P/Q = 5a + 10d
= 5 (a + 2d) ✔

a + 2d = $5k^2$, where k ∈ **W** ✔

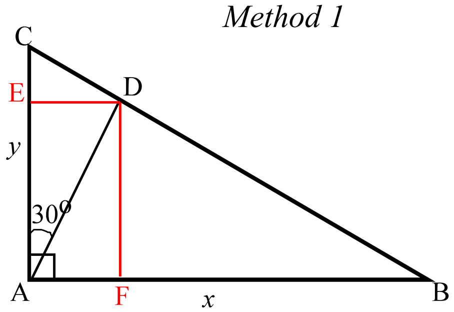
product of all 5 terms =PQ

PQ = P/Q . Q^2 ✔

= 5(a + 2d) . Q^2

= 5 . $5k^2$ .Q^2 ✔

= $(5k.Q)^2$ ✔ or $[5k(a+d)(a+3d)]^2$ [5]
This is a perfect square as required.

10. 

Method 1 Draw in the altitudes of △ACD and △ADB that pass through D
Let intersection of altitude and AC be E and intersection of the altitude and AB be F.

ED = AD sin 30° and FD = AD sin 60°

= ½ AD $= \frac{\sqrt{3}}{2}AD$ ✔

Area △ACD + △ADB = △ABC

$\frac{1}{2}y \cdot \frac{1}{2}AD + \frac{1}{2}x \cdot \frac{\sqrt{3}}{2}AD = \frac{1}{2}xy$ ✔

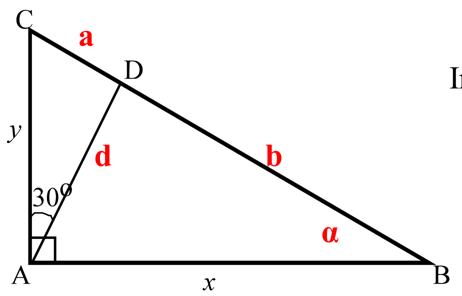
$\left(\frac{1}{2}y + \frac{\sqrt{3}}{2}x\right)AD = xy$ ✔

$\left(y + \sqrt{3}x\right)AD = 2xy$ ✔

$AD = \frac{2xy}{\left(\sqrt{3}x + y\right)}$ ✔ [5]

Method 2

Let AD = d, CD= a , BD = b and ABC =α



In △ABC $\frac{d}{\sin\alpha} = \frac{b}{\sin 60} \Rightarrow b = \frac{\sqrt{3}\,d}{2\sin\alpha}$ ✔

$$\text{and } \sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$$

$$b = \frac{\sqrt{3}\,d\sqrt{x^2+y^2}}{2y} \quad ① ✔$$

In △ACD $\frac{d}{\sin(90-\alpha)} = \frac{a}{\sin 30} \Rightarrow a = \frac{d}{2\cos\alpha}$

$$\text{and } \cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$$

$$a = \frac{d\sqrt{x^2+y^2}}{2x} \quad ② ✔$$

Length of CD gives $\sqrt{x^2+y^2} = a + b$

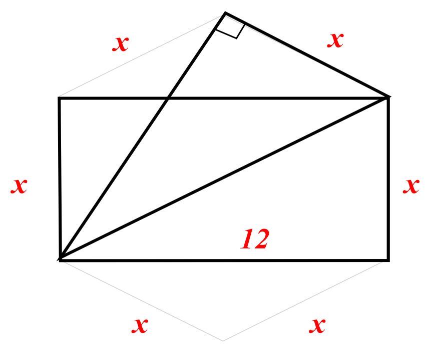
$$\sqrt{x^2+y^2} = \frac{d\sqrt{x^2+y^2}}{2}\left(\frac{1}{x} + \frac{\sqrt{3}}{y}\right) ✔$$

$$1 = \frac{d}{2}\left(\frac{1}{x} + \frac{\sqrt{3}}{y}\right) ✔$$

$$d = \frac{2xy}{\sqrt{3}\,x + y}$$

[5]

11. *Method 1*



Complete a regular hexagon by adding two sides of length x cm below the rectangle.

Since the diagonal of the rectangle can be found using Pythagoras and then use the fact that the diagonal will equal the diameter of the circle (angle in semi circle is 90°) enclosing the regular hexagon. ✔✔

Diameter of hexagon circle = $2x$

$$2x = \sqrt{12^2 + x^2} ✔$$

$$4x^2 = 144 + x^2$$

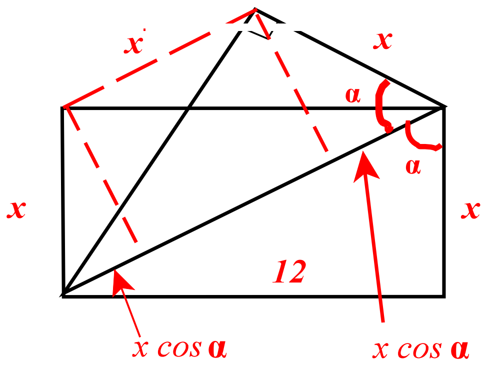
$$3x^2 = 144$$

$$x = 4\sqrt{3}$$

✔✔

[5]

Method 2



Diagonal $\sqrt{x^2 + 144} = x + 2x\cos\alpha$ ✔

$$= x + \frac{2x.x}{\sqrt{x^2 + 144}}$$ ✔

$$\Rightarrow \quad x^2 + 144 = x\sqrt{x^2 + 144} + 2x^2$$

$$44 - x^2 = x\sqrt{x^2 + 144}$$ ✔

$$20736 - 288x^2 + x^4 = 144x^2 + x^4$$

$$20736 = 432x^2$$

$$x^2 = 48$$

$$x = 4\sqrt{3}$$

✔✔ [5]